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Faculty of Engineering
Industrial Engineering Department

Course :
Quantitative Methods (65211)

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Chapter 3

Discrete Random Variables and Probability Distributions

3-1 DISCRETE RANDOM VARIABLES:

➤ Example:

A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used.

X : denote the number of lines in use. Then, X can assume any of the integer values 0 through 48.

When the system is observed, if 10 lines are in use, $\mathbf{x} = 10$.

➤ Example:

In a semiconductor manufacturing process, two wafers from a lot are tested. Each wafer is classified as *pass* or *fail*. Assume that the probability that a wafer passes the test is 0.8 and that wafers are independent.

The random variable X is defined to be equal to the number of wafers that pass.

| Outcome | | Probability | x |
|---------|---------|-------------|-----|
| Wafer 1 | Wafer 2 | | |
| Pass | Pass | 0.64 | 2 |
| Fail | Pass | 0.16 | 1 |
| Pass | Fail | 0.16 | 1 |
| Fail | Fail | 0.04 | 0 |

$$P(pf) = 0.8(0.2) = 0.16$$

3-2 PROBABILITY DISTRIBUTION AND PROBABILITY MASS FUNCTION:

❖ Random variables are so important in random experiments that sometimes we essentially ignore the original sample space of the experiment and focus on the probability distribution of the random variable.

❖ The *probability distribution* of a random variable X is a description of the probabilities associated with the possible values of X . For a discrete random variable, the distribution is

➤ **Example:**

There is a chance that a bit transmitted through a digital transmission channel is received in error.

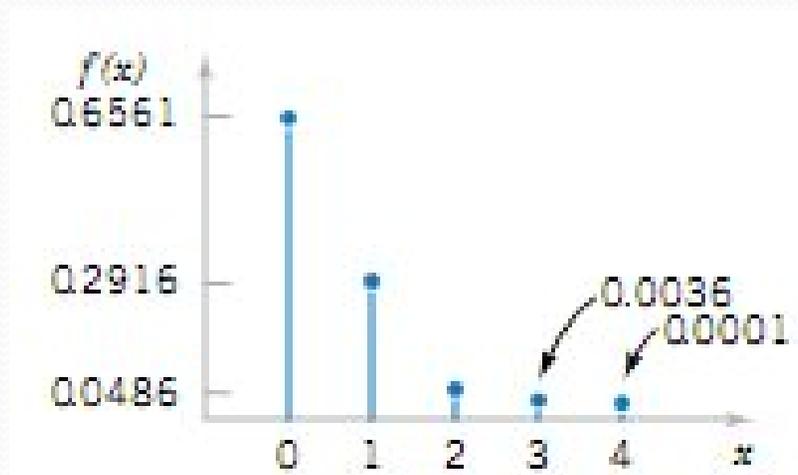
Let X equal the number of bits in error in the next four bits transmitted

The Possible Values of X are $\{0,1,2,3,4\}$

Suppose That:

$$P(X = 0) = 0.6561 \quad P(X = 1) = 0.2916 \quad P(X = 2) = 0.0486$$

$$P(X = 3) = 0.0036 \quad P(X = 4) = 0.0001$$



Probability Distribution

❖ Probability Mass Function:

For a discrete random variable X with possible values x_1, x_2, \dots, x_n , a **probability mass function** is a function such that

$$(1) \quad f(x_i) \geq 0$$

$$(2) \quad \sum_{i=1}^n f(x_i) = 1$$

$$(3) \quad f(x_i) = P(X = x_i)$$

➤ **Example:**

Let the random variable X denote the number of semiconductor wafers that need to be analyzed in order to detect a large particle of contamination.

$P(\text{Large Contamination}) = 0.01$ and that the wafers are **independent**.

Determine the probability distribution of X .

$s = \{p, ap, aap, aaap, aaaap, aaaaaap, \text{ and so forth}\}$

$$P(X=1) = P(p) = 0.01$$

$$P(X=2) = P(ap) = 0.99(0.01) = 0.0099$$

A general formula is

$$P(X = x) = \underbrace{P(aa \dots ap)}_{(x-1)a\text{'s}} = 0.99^{x-1} (0.01), \quad \text{for } x = 1, 2, 3, \dots$$

➤ Question 3-15.

| | | | | | |
|--------|-----|-----|-----|-----|-----|
| x | -2 | -1 | 0 | 1 | 2 |
| $f(x)$ | 1/8 | 2/8 | 2/8 | 2/8 | 1/8 |

a) $P(X \leq 2) = (1/8) + (2/8) + (2/8) + (2/8) + (1/8) = 1$

b) $P(X > -2) = (2/8) + (2/8) + (2/8) + (2/8) + (1/8) = 1 - (1/8) = 7/8$

c) $P(-1 \leq X \leq 1) = (2/8) + (2/8) + (2/8) = 6/8$

d) $P(X \leq -1 \text{ or } X = 2) = (1/8) + (2/8) + (1/8) = 4/8$

3-3 CUMULATIVE DISTRIBUTION FUNCTION:

➤ Example:

There is a chance that a bit transmitted through a digital transmission channel is received in error.

Let X equal the number of bits in error in the next four bits transmitted

The Possible Values of X are $\{0,1,2,3,4\}$

We might be interested in the probability of three or fewer bits being in error.

This question can be expressed as $P(X \leq 3)$

$$\begin{aligned} P(X \leq 3) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &= 0.6561 + 0.2916 + 0.0486 + 0.0036 = 0.9999 \end{aligned}$$

This approach can also be used to determine

$$P(X = 3) = P(X \leq 3) - P(X \leq 2) = 0.0036$$

❖ it is sometimes useful to be able to provide **cumulative probabilities** such as $P(X \leq x)$ and that such probabilities can be used to find the probability mass function of a random variable.

❖ Using cumulative probabilities is an alternate method of describing the probability distribution of a random variable.

In general, for any discrete random variable with possible values x_1, x_2, \dots, x_n , the events $\{X = x_1\}, \{X = x_2\}, \dots, \{X = x_n\}$ are mutually exclusive. Therefore, $P(X \leq x) = \sum_{x_i \leq x} f(x_i)$.

❖ Cumulative Distribution Function:

The **cumulative distribution function** of a discrete random variable X , denoted as $F(x)$, is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

For a discrete random variable X , $F(x)$ satisfies the following properties.

- (1) $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$
- (2) $0 \leq F(x) \leq 1$
- (3) If $x \leq y$, then $F(x) \leq F(y)$

❖ Like a probability mass function, a cumulative distribution function provides probabilities.

❖ Notice that even if the random variable X can only assume integer values, the cumulative distribution function can be defined at no integer values.

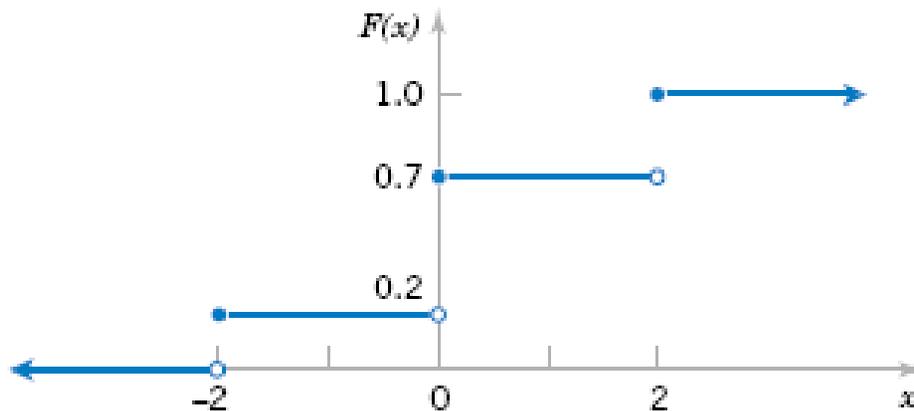
❖ For the last example:

$$F(1.5) = P(X \leq 1.5) = P(X = 0) + P(X = 1) = 0.6561 + 0.2916 = 0.9477$$

➤ **Example:**

Determine the probability mass function of X from the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.2 & -2 \leq x < 0 \\ 0.7 & 0 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$



Cumulative distribution function

$$f(-2) = 0.2 - 0 = 0.2$$

$$f(0) = 0.7 - 0.2 = 0.5$$

$$f(2) = 1.0 - 0.7 = 0.3$$

➤ **Example:**

❖ Suppose that a day's production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch.

Let the random variable X equal the number of nonconforming parts in the sample. What is the cumulative distribution function of X ?

$$P(X = 0) = \frac{800}{850} \cdot \frac{799}{849} = 0.886$$

$$P(X = 1) = 2 \cdot \frac{800}{850} \cdot \frac{50}{849} = 0.111$$

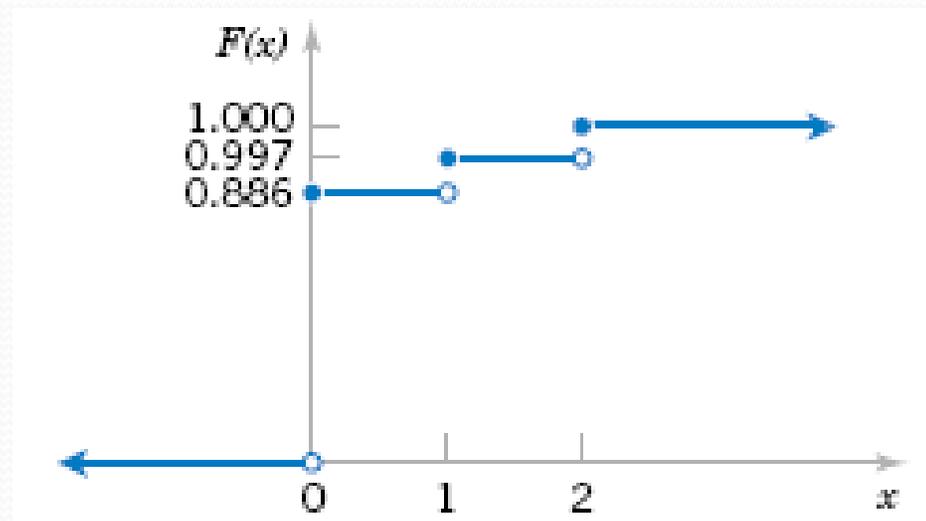
$$P(X = 2) = \frac{50}{850} \cdot \frac{49}{849} = 0.003$$

Therefore,

$$F(0) = P(X \leq 0) = 0.886$$

$$F(1) = P(X \leq 1) = 0.886 + 0.111 = 0.997$$

$$F(2) = P(X \leq 2) = 1$$



➤ Question 3-35.

$$F(x) = \begin{cases} 0 & x < 1 \\ 0.5 & 1 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

a) $P(X \leq 3) = 1$

b) $P(X \leq 2) = 0.5$

c) $P(1 \leq X \leq 2) = 0.5$

d) $P(X > 2) = 1 - P(X \leq 2) = 1 - 0.5 = 0.5$

3-4 MEAN AND VARIANCE OF A DISCRETE RANDOM VARIABLE:

- ❖ **The mean** is a measure of the center or middle of the probability distribution.
- ❖ **The variance** is a measure of the dispersion, or variability in the distribution.

The **mean** or **expected value** of the discrete random variable X , denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_x xf(x)$$

The **variance** of X , denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X - \mu)^2 = \sum_x (x - \mu)^2 f(x) = \sum_x x^2 f(x) - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

❖ Two Different Distribution can have the same mean and variance. Still, these measures are simple, useful summaries of the probability distribution of X .



Equal means and different variances



Equal means and variances

❖ See the derivation of $V(X)$. Page 75.

➤ **Example:**

Let X equal the number of bits in error in the next four bits transmitted.

$$P(X = 0) = 0.6561 \quad P(X = 1) = 0.2916 \quad P(X = 2) = 0.0486$$

$$P(X = 3) = 0.0036 \quad P(X = 4) = 0.0001$$

$$\begin{aligned} \mu = E(X) &= 0f(0) + 1f(1) + 2f(2) + 3f(3) + 4f(4) \\ &= 0(0.6561) + 1(0.2916) + 2(0.0486) + 3(0.0036) + 4(0.0001) \\ &= 0.4 \end{aligned}$$

Although X never assumes the value 0.4, the **weighted average** of the possible values is 0.4.

| x | $x - 0.4$ | $(x - 0.4)^2$ | $f(x)$ | $f(x)(x - 0.4)^2$ |
|-----|-----------|---------------|--------|-------------------|
| 0 | -0.4 | 0.16 | 0.6561 | 0.104976 |
| 1 | 0.6 | 0.36 | 0.2916 | 0.104976 |
| 2 | 1.6 | 2.56 | 0.0486 | 0.124416 |
| 3 | 2.6 | 6.76 | 0.0036 | 0.024336 |
| 4 | 3.6 | 12.96 | 0.0001 | 0.001296 |

$$V(X) = \sigma^2 = \sum_{i=1}^5 f(x_i)(x_i - 0.4)^2 = 0.36$$

→ Example:

Two new product designs are to be compared on the basis of revenue potential.

The revenue from design A can be predicted quite accurately to be \$3 million.

Marketing concludes that there is a probability of 0.3 that the revenue from design B will be \$7 million, but there is a 0.7 probability that the revenue will be only \$2 million.

Which design do you prefer?

X: denote the revenue from design A.

Y: denote the revenue from design B.

$$E(X) = 3\$$$

$$E(Y) = 0.3(7) + 0.7(2) = 3.5\$$$

We might prefer design B.

For design B:

$$\begin{aligned}\sigma^2 &= (7 - 3.5)^2(0.3) + (2 - 3.5)^2(0.7) \\ &= 5.25 \text{ millions of dollars squared}\end{aligned}$$

$$\text{S.D.} = 2.29\$$$

➤ **Example:**

| | | | | | | |
|---------------------------------|------|------|------|------|------|------|
| $x = \text{number of messages}$ | 10 | 11 | 12 | 13 | 14 | 15 |
| $f(x)$ | 0.08 | 0.15 | 0.30 | 0.20 | 0.20 | 0.07 |

$$E(X) = 10(0.08) + 11(0.15) + \cdots + 15(0.07) = 12.5$$

$$V(X) = 10^2(0.08) + 11^2(0.15) + \cdots + 15^2(0.07) - 12.5^2 = 1.85$$

$$\sigma = \sqrt{V(X)} = \sqrt{1.85} = 1.36$$

❖ **Expected Value of a Function of a Discrete Random Variable:**

If X is a discrete random variable with probability mass function $f(x)$,

$$E[h(X)] = \sum_x h(x)f(x)$$

➤ **Example:**

X is the number of bits in error in the next four bits transmitted.

What is the Expected Value of the square of the number of bits in error??

$$h(X) = X^2$$

Therefore,

$$E[h(X)] = 0^2 \times 0.6561 + 1^2 \times 0.2916 + 2^2 \times 0.0486 \\ + 3^2 \times 0.0036 + 4^2 \times 0.0001 = 0.52$$

❖ In a special case that $h(X) = aX + b$ a, b: **constants**

$$E[h(X)] = a E(X) + b.$$

➤ **Question 3-47.**

The range of the random variable X is $[0, 1, 2, 3, x]$ where x is unknown. If each value is equally likely and the mean of X is 6, determine x .

$$f(x_i) = 1/5$$

$$0 f(0) + 1 f(1) + 2 f(2) + 3 f(3) + x f(x) = 6$$

$$0(1/5) + 1(1/5) + 2(1/5) + 3(1/5) + x(1/5) = 6$$

$$x = 24$$

3-5 DISCRETE UNIFORM DISTRIBUTION:

A random variable X has a **discrete uniform distribution** if each of the n values in its range, say, x_1, x_2, \dots, x_n , has equal probability. Then,

$$f(x_i) = 1/n$$

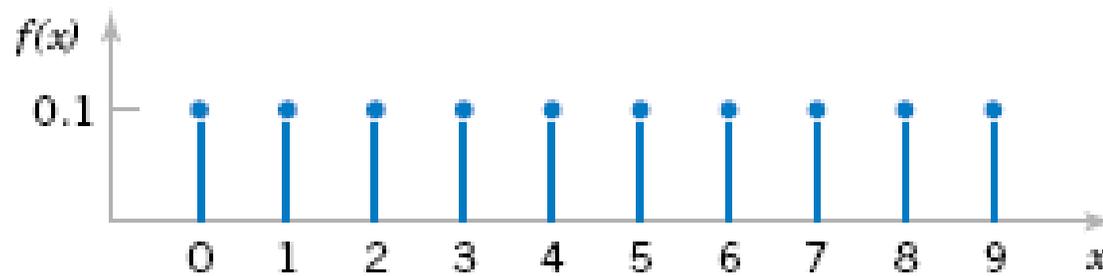
➤ **Example:**

The first digit of a part's serial number is equally likely to be any one of the digits 0 through 9.

If one part is selected from a large batch and X is the first digit of the serial number

X has a discrete uniform distribution with probability 0.1 for each value in $R = \{0, 1, 2, \dots, 9\}$

$$f(x) = 0.1$$



Suppose X is a discrete uniform random variable on the consecutive integers $a, a + 1, a + 2, \dots, b$, for $a \leq b$. The mean of X is

$$\mu = E(X) = \frac{b + a}{2}$$

The variance of X is

$$\sigma^2 = \frac{(b - a + 1)^2 - 1}{12}$$

➤ **Example:**

let the random variable X denote the number of the 48 voice lines that are in use at a particular time.

X is a discrete uniform random variable with a range of 0 to 48.

$$E(X) = (48 + 0)/2 = 24$$

$$\sigma = \{[(48 - 0 + 1)^2 - 1]/12\}^{1/2} = 14.14$$

Let the random variable Y denote the proportion of the 48 voice lines that are in use at a particular time,

$$Y = X/48$$

$$E(Y) = E(X)/48 = 0.5$$

$$V(Y) = V(X)/48^2 = 0.087$$

➤ **Question 3-55.**

Product codes of 2, 3, or 4 letters are equally likely. What is the mean and standard deviation of the number of letters in 100 codes?

It is a uniform distribution

$$\mathbf{E(X) \text{ of } 100 \text{ codes} = 100 [(4+2)/2] = 300}$$

$$\mathbf{S.D. \text{ of } 100 \text{ codes} = (100)^2 * [(4-2+1)^2 - 1]/12 = 6666.67}$$

3-6 BINOMIAL DISTRIBUTION:

❖ Consider the following:

1. Flip a coin 10 times. Let X = number of heads obtained.
2. A machine produces 1% defective parts. Let X = number of defective parts in the next 25 parts produced.
3. In the next 20 births at a hospital. Let X = the number of female births.

❖ Each of these random experiments can be thought of as consisting of a series of repeated, random trials.

❖ The random variable in each case is a count of the number of trials that meet a specified criterion.

❖ The outcome from each trial either meets the criterion that X counts or it does not; (Success / Failure)  **LABELS**

❖ A trial with only two possible outcomes is used so frequently as a building block of a random experiment that it is called a **Bernoulli trial**.

❖ It is usually assumed that the trials that constitute the random experiment are **independent**. (probability of a success in each trial is constant)

➤ **Example:**

The chance that a bit transmitted through a digital transmission channel is received in error is 0.1.

E: a bit in error

O: a bit is okay.

| Outcome | x | Outcome | x |
|-------------|-----|-------------|-----|
| <i>OOOO</i> | 0 | <i>E000</i> | 1 |
| <i>OOOE</i> | 1 | <i>EOOE</i> | 2 |
| <i>OOEO</i> | 1 | <i>EOEO</i> | 2 |
| <i>OOEE</i> | 2 | <i>EOEE</i> | 3 |
| <i>OEOO</i> | 1 | <i>EEOO</i> | 2 |
| <i>OEOE</i> | 2 | <i>EEOE</i> | 3 |
| <i>OEE0</i> | 2 | <i>EEEE</i> | 3 |
| <i>OEEE</i> | 3 | <i>EEEE</i> | 4 |

The event that $X=2$ consists of the six outcomes:

$$\{EEOO, EOEO, EOOE, OEE O, OEEO, OOE E\}$$

Using the assumption that the trials are independent,

$$P(EEOO) = P(E)P(E)P(O)P(O) = (0.1)^2(0.9)^2 = 0.0081$$

$$P(X = 2) = 6(0.0081) = 0.0486$$

In general,

$$P(X = x) = (\text{number of outcomes that result in } x \text{ errors}) \text{ times } (0.1)^x(0.9)^{4-x}$$

$$P(X = x) = \binom{4}{x}(0.1)^x(0.9)^{4-x}$$

❖ Binomial Distribution:

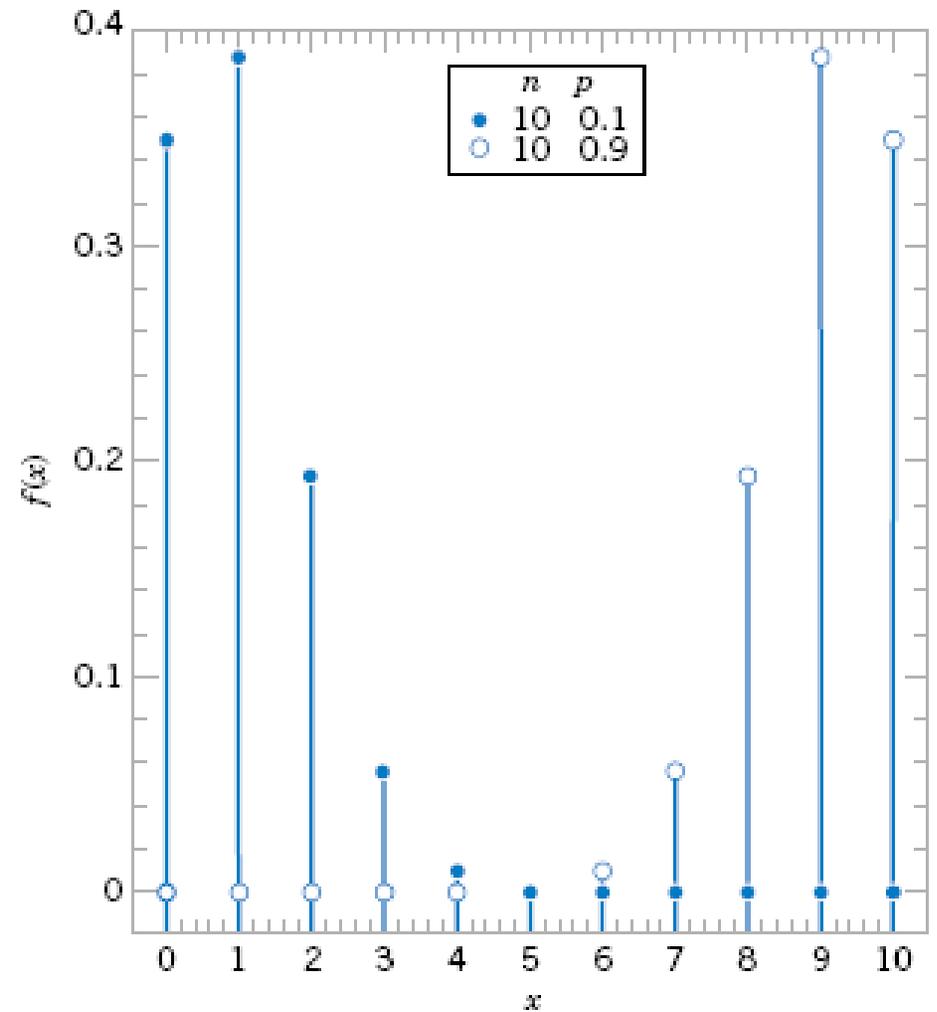
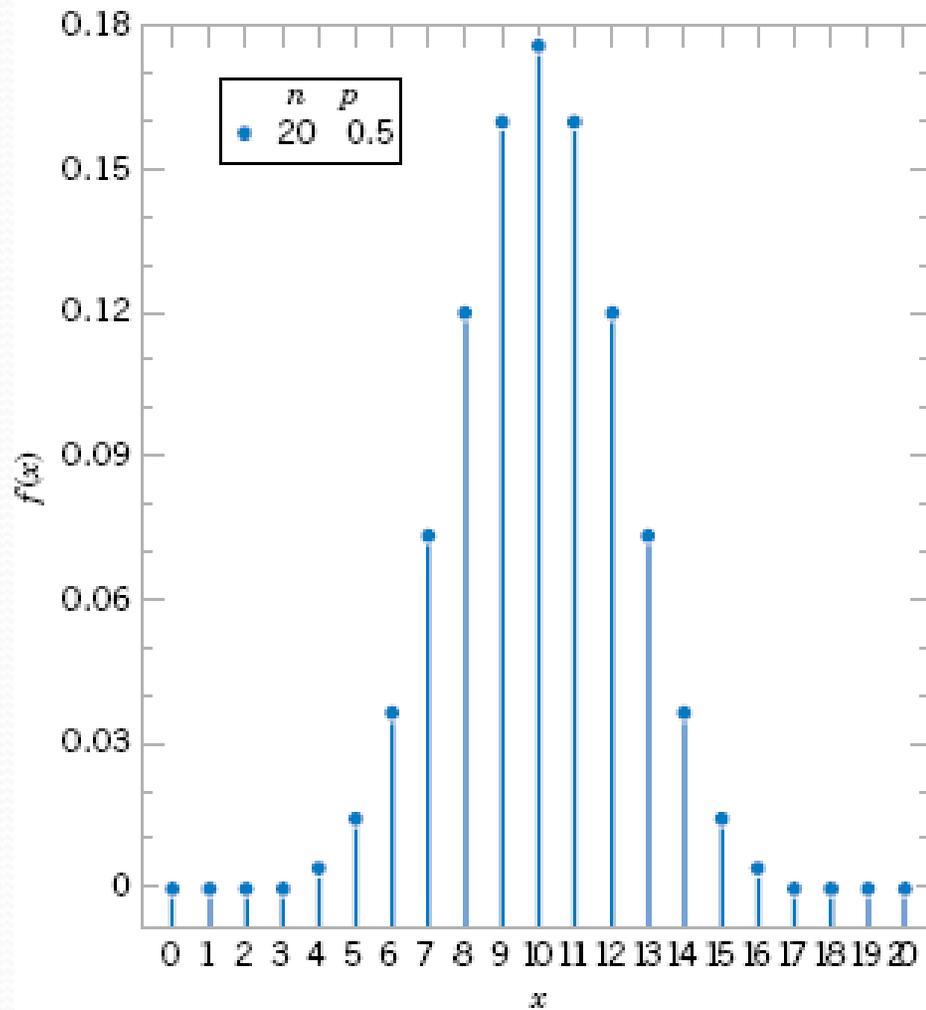
A random experiment consists of n Bernoulli trials such that

- (1) The trials are independent
- (2) Each trial results in only two possible outcomes, labeled as “success” and “failure”
- (3) The probability of a success in each trial, denoted as p , remains constant

The random variable X that equals the number of trials that result in a success has a **binomial random variable** with parameters $0 < p < 1$ and $n = 1, 2, \dots$. The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1 - p)^{n-x} \quad x = 0, 1, \dots, n$$

❖ Binomial Distribution Examples:



➤ **Example:**

Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Find the probability that in the next 18 samples, exactly 2 contain the pollutant.

$$P(X = 2) = \binom{18}{2} (0.1)^2 (0.9)^{16}$$

$$P(X = 2) = 153(0.1)^2(0.9)^{16} = 0.284$$

Determine the probability that at least four samples contain the pollutant.

$$P(X \geq 4) = \sum_{x=4}^{18} \binom{18}{x} (0.1)^x (0.9)^{18-x}$$

$$\begin{aligned} P(X \geq 4) &= 1 - P(X < 4) = 1 - \sum_{x=0}^3 \binom{18}{x} (0.1)^x (0.9)^{18-x} \\ &= 1 - [0.150 + 0.300 + 0.284 + 0.168] = 0.098 \end{aligned}$$

Determine the probability that $3 \leq X < 7$.

$$\begin{aligned} P(3 \leq X < 7) &= \sum_{x=3}^6 \binom{18}{x} (0.1)^x (0.9)^{18-x} \\ &= 0.168 + 0.070 + 0.022 + 0.005 \\ &= 0.265 \end{aligned}$$

❖ **Binomial Distribution (Mean and Variance):**

If X is a binomial random variable with parameters p and n ,

$$\mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1 - p)$$

➤ **Example:**

For the last example: $E(X) = 18(0.1) = 1.8$

$V(X) = 18(0.1)(1-0.1) = 1.62$

➤ Question 3-71.

The phone lines to an airline reservation system are occupied 40% of the time. Assume that the events that the lines are occupied on successive calls are independent. Assume that 10 calls are placed to the airline.

A- What is the probability that for exactly three calls the lines are occupied?

$$P(X = 3) = [10!/(3!7!)] * (0.4)^3 * (0.6)^7 = 0.215$$

B- What is the probability that for at least one call the lines are not occupied?

$$\text{Probability} = 1 - P(X = 0) = 1 - \{ [10!/(0!10!)] * (0.4)^0 * (0.6)^{10} \} = 1 - 0.215 = 1 - 0.006 = 0.994$$

C- What is the expected number of calls in which the lines are all occupied?

$$E(X) = 10(0.4) = 4$$

3-7 GEOMETRIC AND NEGATIVE BINOMIAL DISTRIBUTION:

3-7.1 Geometric Distribution:

- ❖ Assume a series of Bernoulli trials (independent trials with constant probability p of a success on each trial)
- ❖ Trials are conducted until a success is obtained.
- ❖ Let the random variable X denote the number of trials until the first success.

➤ **Example:**

The chance that a bit transmitted through a digital transmission channel is received in error is 0.1.

Assume the transmissions are independent events

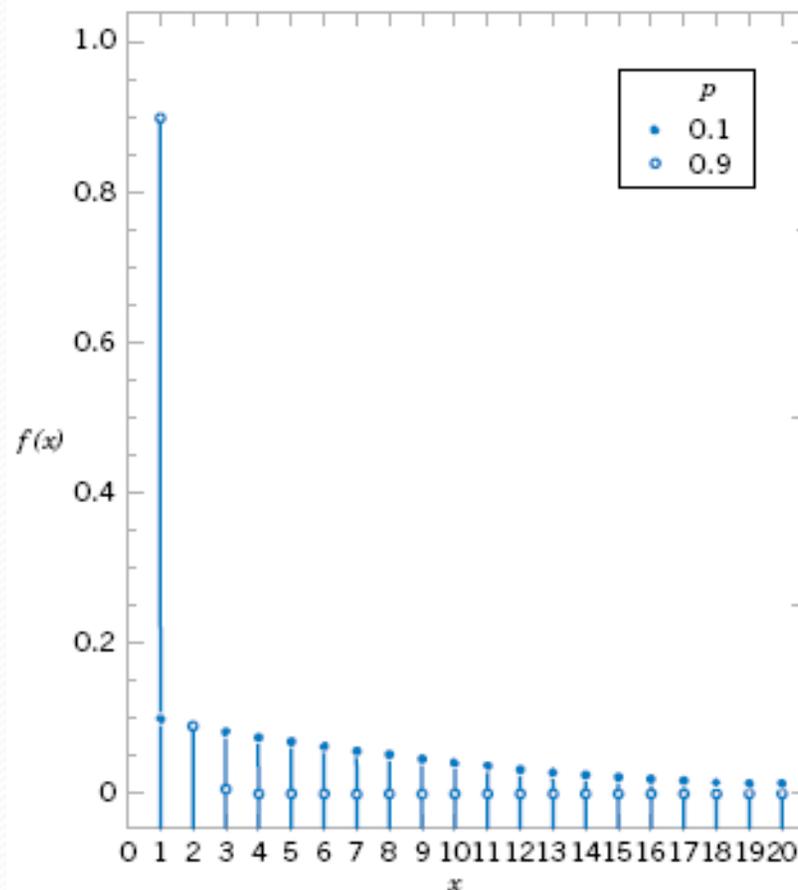
X : # of bits transmitted until the first error occurs.

$$P(X = 5) = P(OOOOE) = (0.9^4)0.1 = 0.066 \quad \text{where } X = \{1,2,3,\dots\}$$

❖ Geometric Distribution:

In a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable X denote the number of trials until the first success. Then X is a **geometric random variable** with parameter $0 < p < 1$ and

$$f(x) = (1 - p)^{x-1}p \quad x = 1, 2, \dots$$



➤ **Example:**

The probability that a wafer contains a large particle of contamination is 0.01. If it is assumed that the wafers are independent, what is the probability that exactly 125 wafers need to be analyzed before a large particle is detected?

$$P = 0.01$$

X: # of samples analyzed until a large particle is detected.

X is a geometric random variable

$$P(X = 125) = (0.99)^{124} 0.01 = 0.0029$$

❖ Geometric Distribution (Mean and Variance):

If X is a geometric random variable with parameter p ,

$$\mu = E(X) = 1/p \quad \text{and} \quad \sigma^2 = V(X) = (1 - p)/p^2$$

❖ Lack of Memory Property

➤ Example:

For the previous example,

$$E(X) = 1/0.01 = 100$$

$$V(X) = (1-0.01)/0.01^2 = 9900$$

3-7.2 Negative Binomial Distribution:

❖ A generalization of a geometric distribution in which the random variable is the number of Bernoulli trials required to obtain r successes results in the Negative Binomial Distribution.

➤ **Example:**

The probability that a bit transmitted through a digital transmission channel is received in error is 0.1.

Assume the transmissions are independent events, and let the random variable X denote the number of bits transmitted *until the fourth error*.

X has a negative binomial distribution with ($r = 4$).

$P(X=10) = \text{?????}$

The probability that exactly three errors occur in the first nine trials is determined from the binomial distribution to be

$$\binom{9}{3} (0.1)^3 (0.9)^6$$

Because the trials are independent, the probability that exactly three errors occur in the first 9 trials and trial 10 results in the fourth error is the product of the probabilities of these two events, namely,

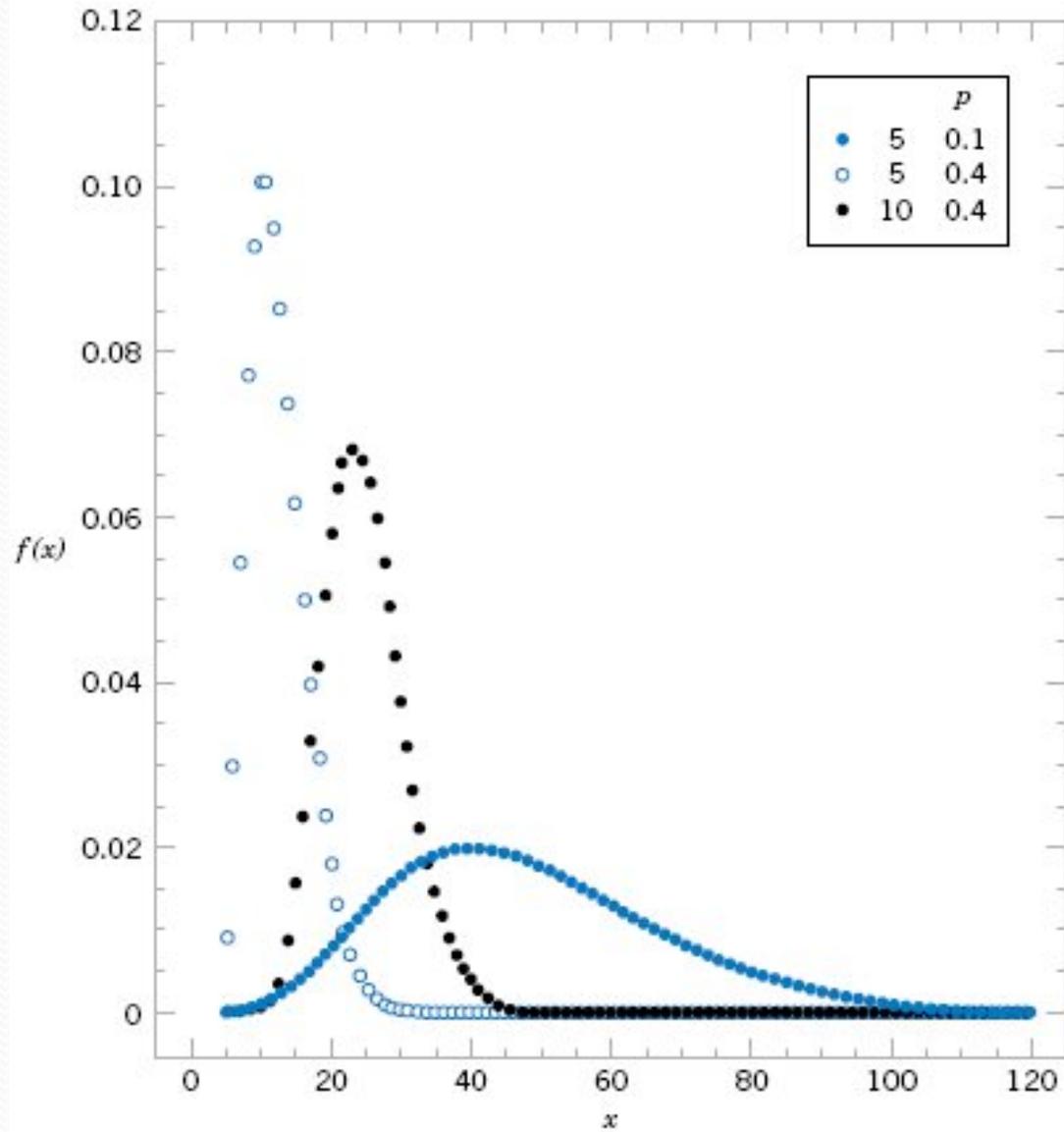
$$\binom{9}{3} (0.1)^3 (0.9)^6 (0.1) = \binom{9}{3} (0.1)^4 (0.9)^6$$

❖ **Negative Binomial Distribution:**

In a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable X denote the number of trials until r successes occur. Then X is a **negative binomial random variable** with parameters $0 < p < 1$ and $r = 1, 2, 3, \dots$, and

$$f(x) = \binom{x-1}{r-1} (1-p)^{x-r} p^r \quad x = r, r+1, r+2, \dots$$

❖ If $r = 1$  Geometric



❖ Negative Binomial random variable can be interpreted as the sum of r geometric random variables

Negative Binomial Distribution (Mean and Variance):

If X is a negative binomial random variable with parameters p and r ,

$$\mu = E(X) = r/p \quad \text{and} \quad \sigma^2 = V(X) = r(1 - p)/p^2$$

➤ Example:

A Web site contains three identical computer servers. Only one is used to operate the site, and the other two are spares. The probability of a failure is 0.0005. Assuming that each request represents an independent trial, what is the mean number of requests until failure of all three servers?

$$E(X) = 3/0.0005 = 6000 \text{ requests}$$

What is the probability that all three servers fail within five requests?

$$\begin{aligned} P(X \leq 5) &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= 0.0005^3 + \binom{3}{2} 0.0005^3 (0.9995) + \binom{4}{2} 0.0005^3 (0.9995)^2 \\ &= 1.25 \times 10^{-10} + 3.75 \times 10^{-10} + 7.49 \times 10^{-10} \\ &= 1.249 \times 10^{-9} \end{aligned}$$

➤ **Question 3-85.**

The probability of a successful optical alignment in the assembly of an optical data storage product is 0.8. Assume the trials are independent.

A) What is the probability that the first successful alignment requires exactly four trials?

Then X is a geometric random variable with $p = 0.8$

$$P(X = 4) = (1 - 0.8)^3 0.8 = 0.2^3 0.8 = 0.0064$$

B) What is the probability that the first successful alignment requires at least four trials?

$$\begin{aligned} P(X \leq 4) &= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) \\ &= (1 - 0.8)^0 0.8 + (1 - 0.8)^1 0.8 + (1 - 0.8)^2 0.8 + (1 - 0.8)^3 0.8 \\ &= 0.8 + 0.2(0.8) + 0.2^2(0.8) + 0.2^3 0.8 = 0.9984 \end{aligned}$$

C) What is the probability that the first successful alignment requires at least four trials?

$$\begin{aligned} P(X \geq 4) &= 1 - P(X \leq 3) = 1 - [P(X = 1) + P(X = 2) + P(X = 3)] \\ &= 1 - [(1 - 0.8)^0 0.8 + (1 - 0.8)^1 0.8 + (1 - 0.8)^2 0.8] \\ &= 1 - [0.8 + 0.2(0.8) + 0.2^2(0.8)] = 1 - 0.992 = 0.008 \end{aligned}$$

3-8 HYPERGEOMETRIC DISTRIBUTION:

❖ 850 manufactured parts contains 50 non conforming to customer requirements.

Select two parts without replacement.

Let A and B denote the events that the first and second parts are nonconforming, respectively.

$$P(B|A) = 49/849 \quad \& \quad P(A) = 50/850$$

Trials are **not independent**.

Let X: the number of nonconforming parts in the sample.

$$P(X = 0) = (800/850) * (799/849) = 0.886$$

$$P(X = 1) = (800/850) * (50/849) + (50/850) * (800/849) = 0.11$$

$$P(X = 2) = (50/850) * (49/849) = 0.003$$

❖ A general formula for computing probabilities when samples are selected without replacement is quite useful.

❖ Hypergeometric Distribution:

A set of N objects contains

K objects classified as successes

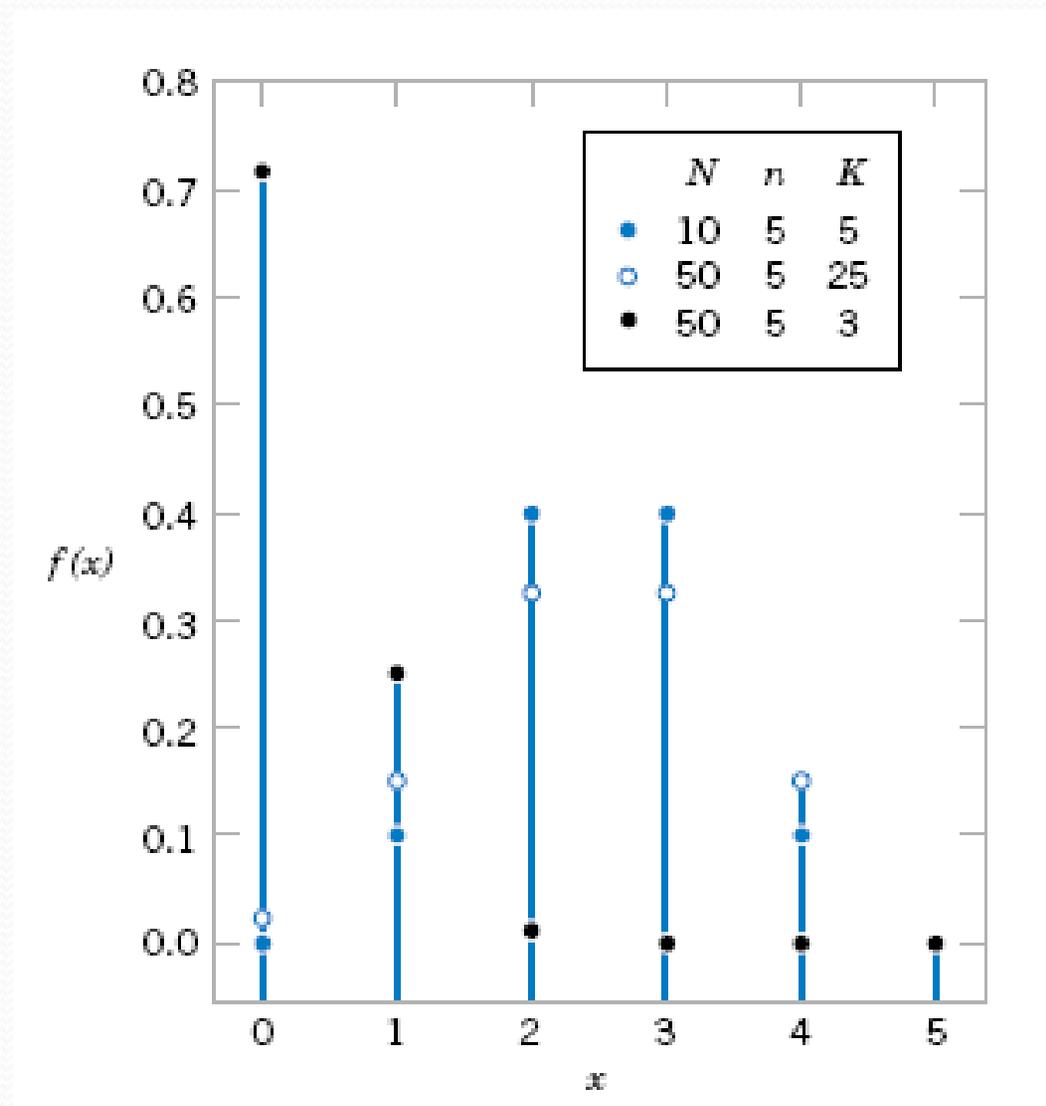
$N - K$ objects classified as failures

A sample of size n objects is selected randomly (without replacement) from the N objects, where $K \leq N$ and $n \leq N$.

Let the random variable X denote the number of successes in the sample. Then X is a **hypergeometric random variable** and

$$f(x) = \frac{\binom{K}{x} \binom{N-K}{n-x}}{\binom{N}{n}} \quad x = \max\{0, n + K - N\} \text{ to } \min\{K, n\}$$

❖ The expression $\{K,n\}$ min is used in the definition of the range of X



➤ **Example:**

Batch, 100 from local supplier, 200 from next state supplier.

4 parts are selected randomly without replacement.

What is the probability they are all from the local supplier????

X: # of parts in the sample from the local supplier.

X has a hypergeometric distribution.

$$P(X = 4) = \frac{\binom{100}{4} \binom{200}{0}}{\binom{300}{4}} = 0.0119$$

What is the probability that two or more parts in the sample are from the local supplier?

$$\begin{aligned} P(X \geq 2) &= \frac{\binom{100}{2}\binom{200}{2}}{\binom{300}{4}} + \frac{\binom{100}{3}\binom{200}{1}}{\binom{300}{4}} + \frac{\binom{100}{4}\binom{200}{0}}{\binom{300}{4}} \\ &= 0.298 + 0.098 + 0.0119 = 0.408 \end{aligned}$$

What is the probability that at least one part in the sample is from the local supplier?

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{100}{0}\binom{200}{4}}{\binom{300}{4}} = 0.804$$

❖ Hypergeometric Distribution (Mean and Variance):

If X is a hypergeometric random variable with parameters N , K , and n , then

$$\mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1-p) \left(\frac{N-n}{N-1} \right)$$

where $p = K/N$.

➤ Example:

For the previous Example,

$$E(X) = 4(100/300) = 1.33$$

$$V(X) = 4(1/3)(2/3)[(300-4)/299] = 0.88$$

❖ If n is small relative to N , the correction is small and the hypergeometric distribution is similar to the binomial.

➤ **Question 3-101.**

A company employs 800 men under the age of 55. Suppose that 30% carry a marker on the male chromosome that indicates an increased risk for high blood pressure.

A) If 10 men in the company are tested for the marker in this chromosome, what is the probability that exactly 1 man has the marker?

X : # of men who carry the marker on the male chromosome for an increased risk for high blood pressure.

$N=800, K=240, n=10$

$$P(X = 1) = \frac{\binom{240}{1} \binom{560}{9}}{\binom{800}{10}} = \frac{\left(\frac{240!}{1!239!}\right) \left(\frac{560!}{9!551!}\right)}{\frac{800!}{10!790!}} = 0.1201$$

B) If 10 men in the company are tested for the marker in this chromosome, what is the probability that more than 1 has the marker?

$$P(X > 1) = 1 - P(X \leq 1) = 1 - [P(X = 0) + P(X = 1)]$$

$$P(X = 0) = \frac{\binom{240}{0} \binom{560}{10}}{\binom{800}{10}} = \frac{\left(\frac{240!}{0!240!}\right) \left(\frac{560!}{10!550!}\right)}{\frac{800!}{10!790!}} = 0.0276$$

$$P(X > 1) = 1 - P(X \leq 1) = 1 - [0.0276 + 0.1201] = 0.8523$$

3-8 POISSON DISTRIBUTION:

➤ Example:

Consider the transmission of n bits over a digital communication channel.

X : # of bits in error.

When the probability that a bit is in error is constant and the transmissions are independent, X has a binomial distribution.

Let $\lambda = np$, then $E(X) = np = \lambda$

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

Suppose that n is **INCREASES** and p is **DECREASES**, S.t. $E(X) = \lambda$ remains **CONSTANT**, then:

➤ Example:

- ✓ Flaws occur at random along the length of a thin copper wire.
- ✓ X : the # of flaws in a length of L millimeters of wire
- ✓ Suppose that the average number of flaws in L millimeters is λ .
- ✓ Partition the length of wire into n subintervals of small length, say, 1 micrometer each.
- ✓ If the subinterval chosen is small enough, the probability that more than one flaw occurs in the subinterval is negligible.
- ✓ We can interpret the assumption that flaws occur at random to imply that every subinterval has the same probability of containing a flaw, say, p .
- ✓ If we assume that the probability that a subinterval contains a flaw is independent of other subintervals, we can model the distribution of X as *approximately a binomial* random variable.

$$E(X) = \lambda = np$$

$$p = \lambda/n$$

✓ That is, the probability that a subinterval contains a flaw is p . With small enough subintervals, n is very large and p is very small. Therefore, the distribution of X is obtained as in the previous example.

✓ The same reasoning can be applied to any interval, including an interval of time, an area, or a volume.

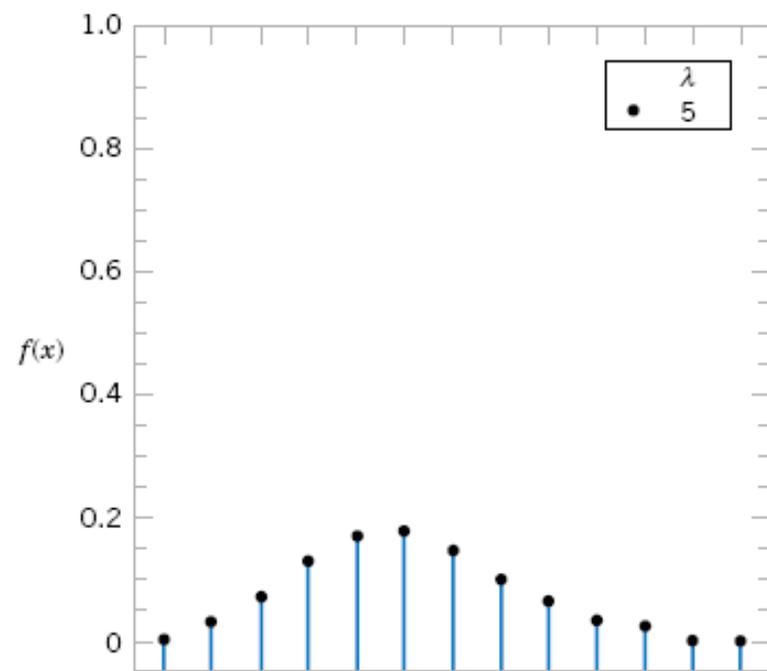
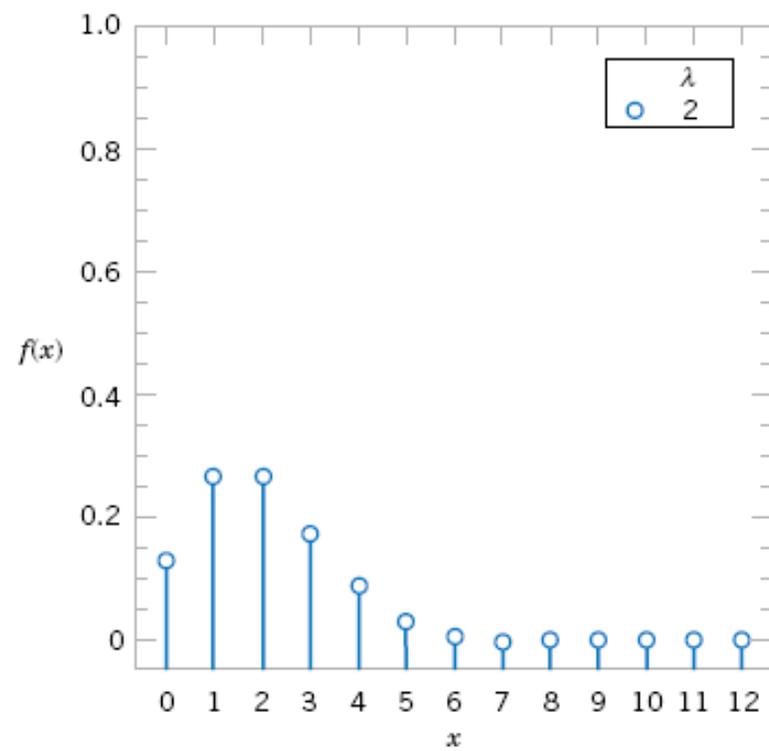
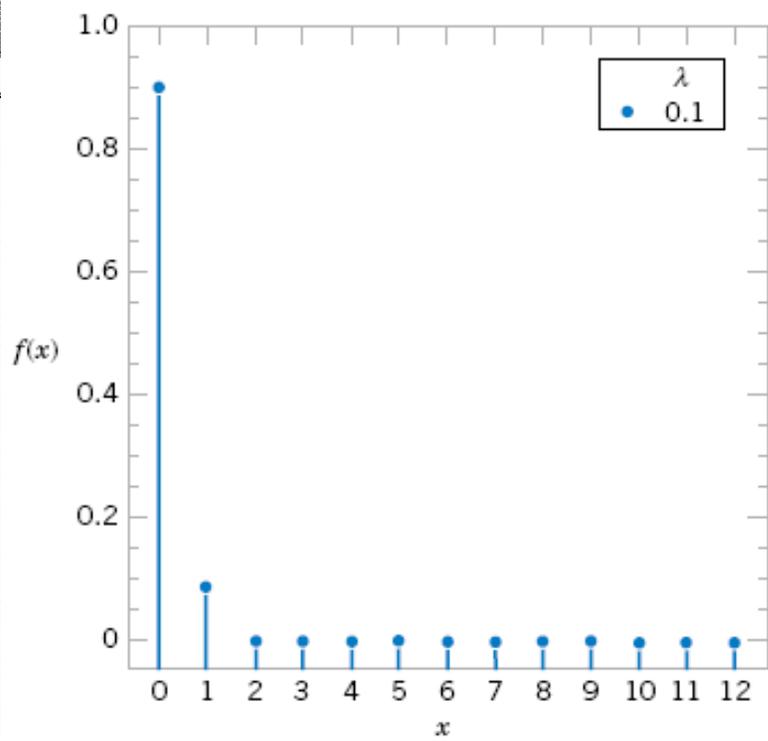
❖ Poisson Distribution:

Given an interval of real numbers, assume counts occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that

- (1) the probability of more than one count in a subinterval is zero,
- (2) the probability of one count in a subinterval is the same for all subintervals and proportional to the length of the subinterval, and
- (3) the count in each subinterval is independent of other subintervals, the random experiment is called a **Poisson process**.

The random variable X that equals the number of counts in the interval is a **Poisson random variable** with parameter $0 < \lambda$, and the probability mass function of X is

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad x = 0, 1, 2, \dots$$



❖ It is important to **use consistent units** in the calculation of probabilities, means, and variances involving Poisson random variables.

- average number of flaws per millimeter of wire is 3.4, then the
- average number of flaws in 10 millimeters of wire is 34, and the
- average number of flaws in 100 millimeters of wire is 340.

❖ **Poisson Distribution (Mean and Variance):**

If X is a Poisson random variable with parameter λ , then

$$\mu = E(X) = \lambda \quad \text{and} \quad \sigma^2 = V(X) = \lambda$$

➤ **Example:**

For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution with a mean of 2.3 flaws per millimeter. Determine the probability of exactly 2 flaws in 1 millimeter of wire.

X: # of flaws in 1 mm of wire.

$E(X) = 2.3$ flaws/mm

$$P(X = 2) = \frac{e^{-2.3} 2.3^2}{2!} = 0.265$$

Determine the probability of 10 flaws in 5 millimeters of wire.

X: # of flaws in 5 mm of wire.

$$E(X) = 5 \text{ mm} \times 2.3 \text{ flaws/mm} = 11.5 \text{ flaws}$$

$$P(X = 10) = e^{-11.5} \frac{11.5^{10}}{10!} = 0.113$$

Determine the probability of at least 1 flaw in 2 millimeters of wire.

$$E(X) = 2 \text{ mm} \times 2.3 \text{ flaws/mm} = 4.6 \text{ flaws}$$

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-4.6} = 0.9899$$

➤ **Question 3-111.**

Astronomers treat the number of stars in a given volume of space as a Poisson random variable. The density in the Milky Way Galaxy in the vicinity of our solar system is one star per 16 cubic light years.

A) What is the Probability of two or more stars in 16 cubic light years?

X : # of stars per 16 cubic light years

$$P(X \geq 2) = 1 - [P(X=0) + P(X=1)] = 0.564$$

B) How many cubic light years of space must be studied so that the probability of one or more stars exceeds 0.95?

In order that $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-\lambda}$ exceed 0.95, we need $\lambda=3$.

Therefore $3 * 16 = 48$ cubic light years of space must be studied.

